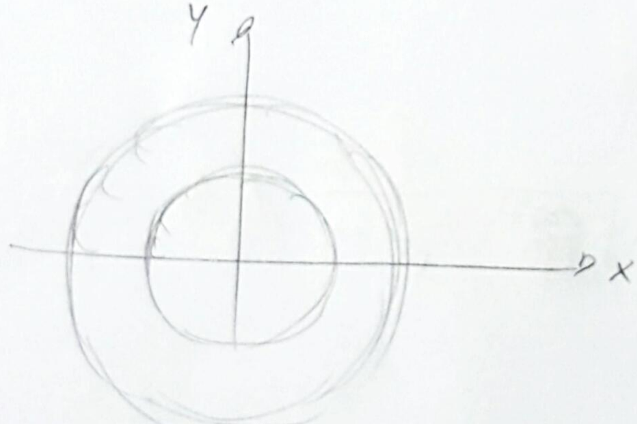
Esercizi integrali di superficie

Es 1

$$\int_\Sigma \frac{1}{z^4}\, d\sigma \qquad \Sigma = \left\{(x,y,z)\in\mathbb{R}^3 : z = \frac{1}{\sqrt{x^2+y^2}},\ 1\le z\le 2\right\}$$

$$\sqrt{x^2+y^2} = \frac{1}{z} \qquad \frac{1}{z} \ge \frac{1}{2} \qquad \frac{1}{z} \le 1$$

$$\frac{1}{2} \le \sqrt{x^2+y^2} \le 1 \Rightarrow \frac{1}{4} \le x^2+y^2 \le 1$$



$$\Phi = \left(x, y, \frac{1}{\sqrt{x^2+y^2}}\right)$$

$$\Phi_x = \left(1, 0, -\frac{\frac{x}{\sqrt{x^2+y^2}}}{x^2+y^2}\right) = \left(1, 0, -\frac{x}{(x^2+y^2)^{\frac{3}{2}}}\right)$$

$$\Phi_y = \left(0,\ 1,\ \frac{-y}{(x^2+y^2)^{\frac{3}{2}}}\right)$$

$$|\Phi_x \wedge \Phi_y| \Rightarrow \left| \begin{pmatrix} i & j & k \\ 1 & 0 & \frac{-x}{(x^2+y^2)^{\frac{3}{2}}} \\ 0 & 1 & \frac{-y}{(x^2+y^2)^{\frac{3}{2}}} \end{pmatrix} \right| = K + \frac{x}{(x^2+y^2)^{\frac{3}{2}}} i + \frac{y}{(x^2+y^2)^{\frac{3}{2}}} j$$

$$\Rightarrow \sqrt{\frac{x^2}{(x^2+y^2)^3} + \frac{y^2}{(x^2+y^2)^3} + 1}$$

$$\iint_D \frac{1}{(x^2+y^2)} \cdot \sqrt{\frac{x^2}{(\ )^6} + \frac{y^2}{(\ )^6} + 1}\ dx\, dy$$

$$\int_0^{2\pi} \int_{1/4}^{1} \rho^3 \cdot \sqrt{\frac{\cos^2\theta}{\rho^4} + \frac{\sin^2\theta}{\rho^4} + 1}\ d\rho\, d\theta$$

$$\frac{2\pi}{4} \int_{1/2}^{1} \underbrace{4\rho^3}_{f'(x)} \underbrace{\sqrt{1+\rho^4}}_{f(x)^\alpha}\ d\rho$$

$$\frac{\pi}{2} \left.\frac{(1+\rho^4)^{\frac{3}{2}}}{\frac{3}{2}}\right|_{\frac{1}{2}}^{1} = \frac{\pi}{3} \left.(1+\rho^4)^{\frac{3}{2}}\right|_{\frac{1}{2}}^{1}$$

Es 6

$$\iint_S \frac{1}{\sqrt{1-y^4}}\, d\sigma$$

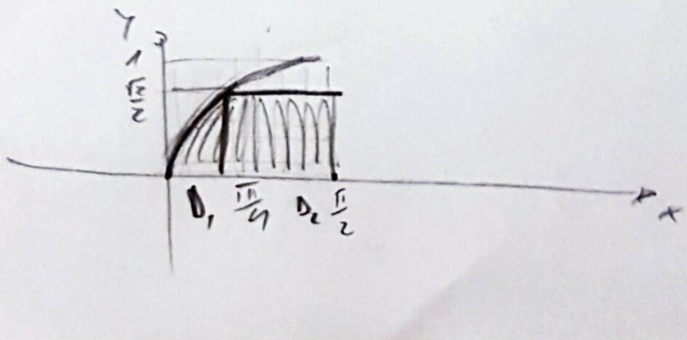
$$S = \left\{ (x,y,z) \in \mathbb{R}^3 : z = x + \frac{\sqrt{2}}{2} y^2,\ 0 \le x \le \frac{\pi}{2},\ 0 \le y \le \frac{\sqrt{2}}{2},\ y \le \sin x \right\}$$

$$\Phi = \left(x, y, x + \frac{\sqrt{2}}{2} y^2 \right)$$

$$\begin{aligned} \Phi_x &= (1, 0, 1) \\ \Phi_y &= (0, 1, \sqrt{2} y) \end{aligned} \Rightarrow \begin{pmatrix} i & j & k \\ 1 & 0 & 1 \\ 0 & 1 & \sqrt{2}y \end{pmatrix} =$$

$$= \underset{\sim}{k} - \sqrt{2} y \underset{\sim}{j} - \underset{\sim}{i} = (-1, -\sqrt{2}y, 1)$$

$$|\Phi_x \wedge \Phi_y| = \sqrt{1 + 2y^2 + 1} = \sqrt{2}\sqrt{1+y^2}$$



$$\sqrt{2}\iint_D \frac{1}{\sqrt{1-y^4}}\sqrt{1+y^2}\,dx\,dy$$

$$\sqrt{2}\iint_D \frac{\cancel{\sqrt{1+y^2}}}{\sqrt{(\cancel{1+y^2})(1-y^2)}}\,dx\,dy$$

$$\sqrt{2}\int_0^{\frac{\pi}{4}} dx \int_0^{\sin x} \frac{1}{\sqrt{1-y^2}}\,dy + \sqrt{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} dx \int_0^{\frac{\sqrt{2}}{2}} \frac{1}{\sqrt{1-y^2}}\,dy$$

$$\sqrt{2}\int_0^{\frac{\pi}{4}} \arcsin(\sin x)\,dx + \sqrt{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\pi}{4}\,dx$$

$$\sqrt{2}\,\frac{x^2}{2}\Big|_0^{\frac{\pi}{4}} + \sqrt{2}\,\frac{\pi}{4}\left(\frac{\pi}{2}-\frac{\pi}{4}\right)$$

$$\frac{\sqrt{2}\,\pi^2}{32} + \frac{\sqrt{2}\,\pi^2}{16} = \frac{3\sqrt{2}\,\pi^2}{32}$$

Es e

$$\int_S z^2\, d\sigma \qquad S = \{(x,y,z) \in \mathbb{R}^3 : z = xy,\ 0<y<\sqrt{3}x,\ x^2+y^2 \le 1$$

$\Phi = (x, y, xy)$

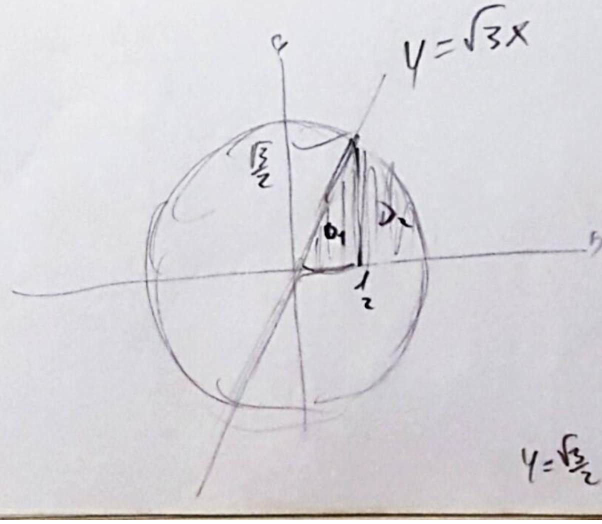
$\Phi_x = (1, 0, y)$

$\Phi_y = (0, 1, x)$ $\Rightarrow \Phi_x \wedge \Phi_y = \begin{pmatrix} i & j & k \\ 1 & 0 & y \\ 0 & 1 & x \end{pmatrix} =$

$= k - xj - yi =$

$|\Phi_x \wedge \Phi_y| = \sqrt{x^2+y^2+1}$ $\qquad = (-y, -x, 1)$

$$\iint x^2y^2\sqrt{x^2+y^2+1}\, dx\, dy$$



$\Theta = \frac{\pi}{3}$

$\begin{cases} x^2+y^2=1 \\ y=\sqrt{3}x \end{cases}$

$x^2+3x^2=1$

$x^2 = \frac{1}{4}$

$y = \frac{\sqrt{3}}{2} \quad x = \frac{1}{2}$

$$\int_0^{\frac{\pi}{3}}\int_0^{4} \rho^5 \cos^2\theta \sin^2\theta \sqrt{\rho^2+1}\, d\rho\, d\theta$$

$$\parallel$$

$$\left(\frac{\sin 2\theta}{2}\right)^2$$

e si integra.